MINISTRY OF HIGHER EDUCATION

& SCIENTIFIC RESEARCH

TIKRIT UNIVERSITY

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AUTOMATIC CONTROL SYSTEMS & MEASUREMENTS

LECTURE NOTES (4TH YEAR – I SEM) (2022-2023)

GHAPTER FOUR

MODELLING A CONTROL SYSTEM

4. INTRODUCTION TO MATHEMATICAL MODELLING

A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately. A mathematical model is not unique for a given system. It is possible to improve the accuracy of a mathematical model by increasing its complexity.

4.1- *Mechanical system:*

In this chapter, let us discuss the differential equation modeling of mechanical systems. There are two types of mechanical systems based on the type of motion.

- Translational mechanical systems
- Rotational mechanical systems

a) Modeling of Translational Mechanical Systems

Translational mechanical systems move along a straight line. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper.

- If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero. Let us now see the force opposed by these three elements individually.

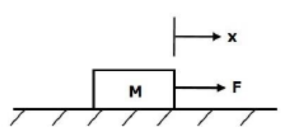
1- Mass

Mass is the property of a body, which stores kinetic energy. If a force is applied on a body having mass M, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.

$$F_m \propto a$$

$$\Rightarrow F_m = Ma = M\frac{d^2x}{dt^2}$$

$$F = F_m = M\frac{d^2x}{dt^2}$$

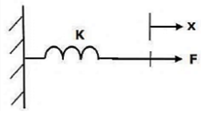


Where,

- F is the applied force
- F_m is the opposing force due to mass
- M is mass
- a is acceleration
- x is displacement

2- Spring

Spring is an element, which stores potential energy. If a force is applied on spring K, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction are negligible.



$$F_k \propto x$$

$$\Rightarrow F_k = Kx$$

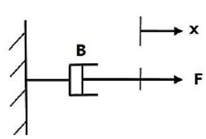
$$F = F_k = Kx$$

Where,

- F is the applied force
- F_k is the opposing force due to elasticity of spring
- K is spring constant
- x is displacement

3- dashpot

If a force is applied on dashpot B, then it is opposed by an opposing force due to friction of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity are negligible.



$$F_b \propto v$$

$$\Rightarrow F_b = Bv = B\frac{dx}{dt}$$

$$F = F_b = B\frac{dx}{dt}$$

Where,

- F_b is the opposing force due to friction of dashpot
- B is the frictional coefficient
- v is velocity
- x is displacement

- *Mass –Spring- Dashpot*

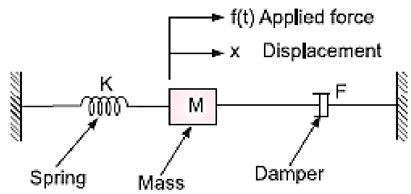
A translational spring-mass-damper system is shown in Fig. below

- Mass –Spring- DashpotThe equation of motion for the system is obtained by applying Newton's second law of translational motion.

Inertia force = Forces acting on the mass.

$$M\frac{d^2x}{dt^2} = -f\frac{dx}{dt} - Kx + f(t)$$

↓ Damping force ↓ Spring restoring force ↓ Applied force



Translational spring-mass damper system.

Rearranging Eq. above, the following equation is obtained

$$M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = f(t) \quad \text{------------------------ (1)}$$

. Assuming initial conditions as zero, taking Laplace transform on both sides of equation (1) following equation is obtained : **M** s^2 X(s) + **f** s X(s) + ***K*** X(s) = **F**(s)

If X(s) is specified as the output and F(s) as input then the transfer function of the system is given by the relation below

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + fs + K}$$

b) Modeling of Rotational Mechanical Systems

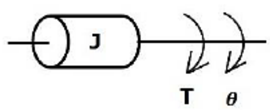
Rotational mechanical systems move about a fixed axis. These systems mainly consist of three basic elements. Those are moment of inertia, torsional spring and dashpot.

If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero. Let us now see the torque opposed by these three elements individually.

- *Moment of Inertia*

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores kinetic energy.

If a torque is applied on a body having moment of inertia J, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.



$$T_j \propto \alpha$$

$$\Rightarrow T_j = J\alpha = J\frac{d^2\theta}{dt^2}$$

$$T = T_j = J\frac{d^2\theta}{dt^2}$$

Where,

- T is the applied torque
- T_j is the opposing torque due to moment of inertia
- J is moment of inertia
- α is angular acceleration
- θ is angular displacement.

- *Torsional Spring*

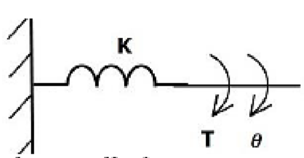
In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores potential energy.

If a torque is applied on torsional spring K, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.

$$T_k \propto \theta$$

$$\Rightarrow T_k = K\theta$$

$$T = T_k = K\theta$$

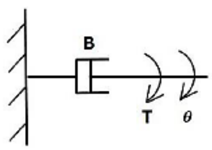


Where,

- T is the applied torque
- T_k is the opposing torque due to elasticity of torsional spring
- K is the torsional spring constant
- θ is angular displacement

- *Torsional Dashpot*

If a torque is applied on dashpot B, then it is opposed by an opposing torque due to the rotational friction of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.



$$T_b \propto \omega$$

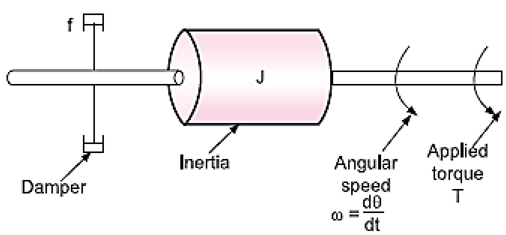
$$\Rightarrow T_b = B\omega = B\frac{d\theta}{dt}$$

$$T = T_b = B\frac{d\theta}{dt}$$

Where,

- T_b is the opposing torque due to the rotational friction of the dashpot
- B is the rotational friction coefficient
- ω is the angular velocity
- θ is the angular displacement.

Consider the system shown below



Rotational inertia damper system.

The system consists of a load inertia and a viscous friction damper. For such a mechanical rotational system, Newton's second law states that

$J\,\alpha = \sum T$

Where J = moment of inertia of the load, kg−m^2

α = angular acceleration of the load, rad/sec^2

T = torque applied to the system, N−m

Applying Newton's second law to the present system, we obtain

$$J\dot{\omega(t)} = -B\omega(t) + T(t)$$

where

J = moment of inertia of the load, kg−m2

b = viscous−friction coefficient, N−m/rad/sec

ω = angular velocity, rad/sec

T = torque, N−m

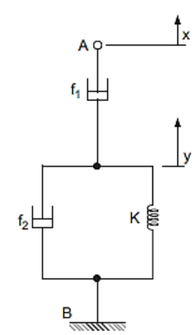
The last equation may be written as $J\,\dot{\omega}(t) + B\,\omega(t) = T$

which is a mathematical model of the mechanical rotational system considered.

The transfer function model for the system can be obtained by taking the Laplace transform of the differential equation, assuming the zero initial condition, and writing the ratio of the output (angular velocity ω) and the input (applied torque T) as follows :

$$J\,s\,\omega(s) + B\,\omega(s) = T(s) \longrightarrow \frac{\omega(s)}{T(s)} = \frac{1}{Js+B}$$

Example -1- Find the transfer function relating displacements y and x for the mechanical system of Fig. below.



Solution. Let force $f(t)$ be applied force at point A. according to Newton's law in mechanical circuit diagram the equations for the system are

$$f_2 \frac{dy}{dt} + Ky + f_1 \frac{d}{dt}(y - x) = 0 \qquad ...(1)$$

$$f_1 \frac{d}{dt}(x - y) = f(t) \qquad ...(2)$$

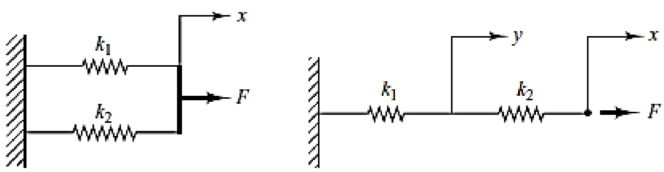
Assuming initial conditions as zero and taking Laplace transform on both sides of
equation (1) following equation is obtained :

$$f_2\, sY(s) + KY(s) + f_1 s[Y(s) - X(s)] = 0 \qquad ...(3)$$

From equation (3), the required transfer function is

$$\frac{Y(s)}{X(s)} = \frac{f_1 s}{[(f_1+f_2)s+K]} = \frac{T_1 s}{(T_1 s+1)} \quad \text{where} \quad T_1 = \frac{f_1}{K} \quad \text{and} \quad T_2 = \frac{(f_1+f_2)}{K}.$$

EXAMPLE -2- Let us obtain the equivalent spring constants for the systems shown in Figures below (a) and (b), respectively.



(a) (b)

Solution. For the springs in parallel [Figure (a)] the equivalent spring constant *k*eq is obtained From

$$k_1 x + k_2 x = F = k_{eq} x$$

or

$$k_{eq} = k_1 + k_2$$

For the springs in series [Figure (b)], the force in each spring is the same. Thus

$$k_1 y = F, \qquad k_2(x - y) = F$$

Elimination of y from these two equations results in

$$k_2\left(x - \frac{F}{k_1}\right) = F$$

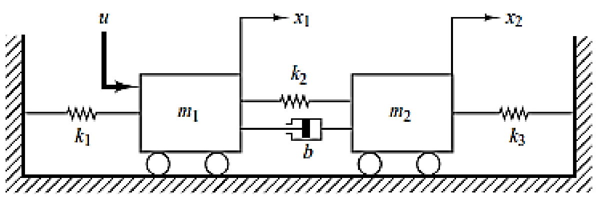
or

$$k_2 x = F + \frac{k_2}{k_1}F = \frac{k_1 + k_2}{k_1}F$$

The equivalent spring constant k_{eq} for this case is then found as

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

EXAMPLE -3- Obtain the transfer functions$X_1(s)$ /U(s) and $X_2(s)$/U(s) of the mechanical system shown in figure below .



Solution. The equations of motion for the system shown in Figure 3–4 are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1s^2 + bs + (k_1 + k_2)]X_1(s) = (bs + k_2)X_2(s) + U(s) \qquad (3–5)$$

$$[m_2s^2 + bs + (k_2 + k_3)]X_2(s) = (bs + k_2)X_1(s) \qquad (3–6)$$

Solving Equation (3–6) for X_2 (s) and substituting it into Equation (3–5) and simplifying, we get

$$[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]X_1(s)$$

$$= (m_2s^2 + bs + k_2 + k_3)U(s)$$

from which we obtain.

From

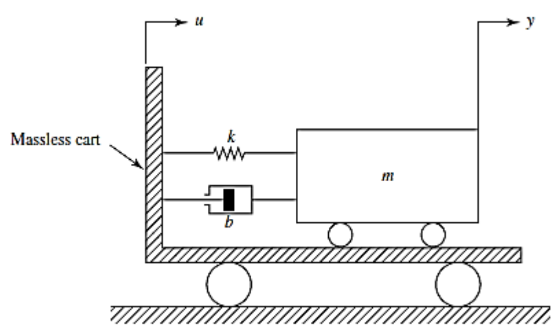
$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \qquad (3–7)$$

Equations (3–6) and (3–7) we have.

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \qquad (3\text{–}8)$$

Equations (3–7) and (3–8) are the transfer functions $X_1(s)$ / U(s) and $X_2(s)$/ U(s), respectively.

EXAMPLE 4 :- Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure below .us obtain mathematical models of this system by assuming that the cart is standing still for t<0and the spring-mass-dashpot system on the cart is also standing still for t<0. In this system, u(t) is the displacement of the cart and is the input to the system. At t=0, the cart is moved at a constant speed. The displacement y(t) of the mass is the output. (The displacement is relative to the ground.) In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant. Obtain the transfer functions Y(s) /U(s) of the mechanical system shown in figure below .



Solution. We assume that the friction force of the dashpot is proportional to and that the spring is a linear spring; that is, the spring force is proportional to y-u.

- For translational systems, Newton's second law states that ma = $\sum F$

Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

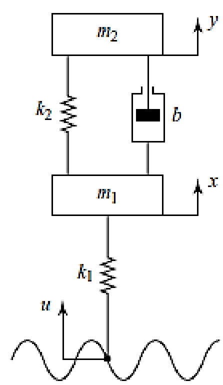
Taking the ratio of Y(s) to U(s), we find the transfer function of the system to be

$$\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

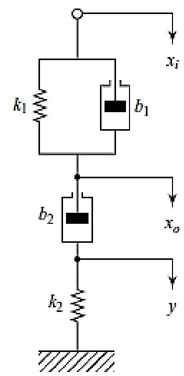
Such a transfer-function representation of a mathematical model is used very frequently in control engineering.

Home work

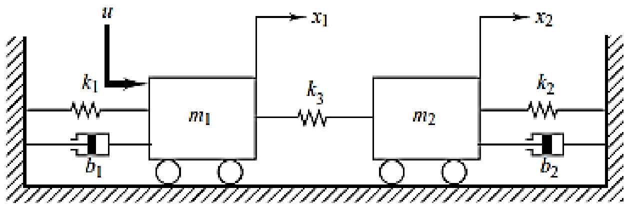
Q.1- Obtain the transfer function Y(s)/U(s) of the system shown in Figure below.



Q.2- Obtain the transfer function Xo(s)/Xi (s) of the mechanical system shown in Figure below .

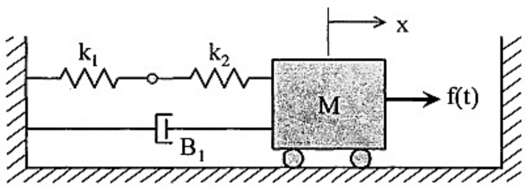


Q.3- Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure below.



Q.4- A Simple Mechanical System find Its Transfer Function if M = 5kg , K_1 = 225 N/m

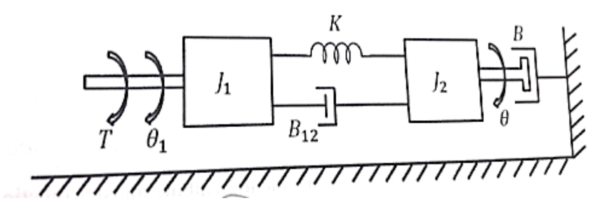
K_2 =150 N/m and B_1=55N.s/m



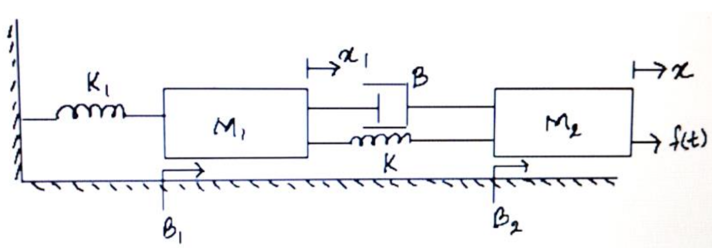
Determine

- the effective spring constant for the system shown above
- Calculate the poles of the system

Q.5- Write the differential equations governing the mechanical rotational system shown below and determine the transfer function.



Q.6- Write the differential equations governing the mechanical translational system as shown in figure below and determine the transfer function.



Q.7- For each of the mechanical systems given in problems 4-1 through 1-5. write the equation of motion in terms of the given mechanical quantities.

4-1.

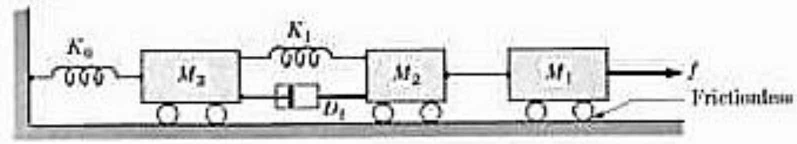


Figure 4-21

4-2.

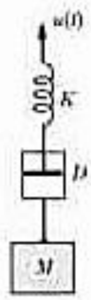


Figure 4-22

4-3.

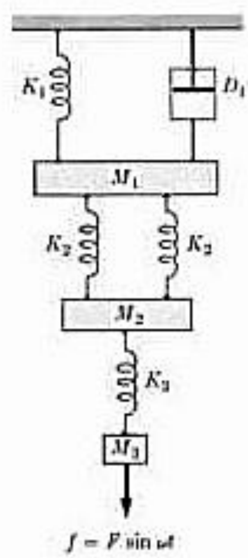


Figure 4-23

4-4.

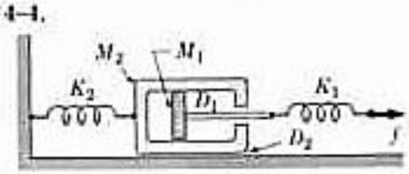


Figure 4-24

4-5.

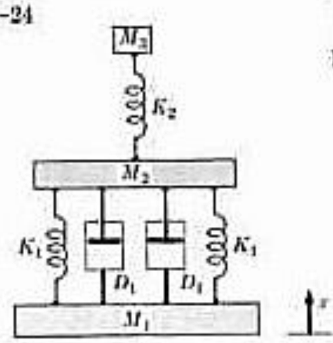


Figure 4-25

Q.7- For each of the mechanical systems given below. write the equation of motion in terms of the given mechanical quantities.

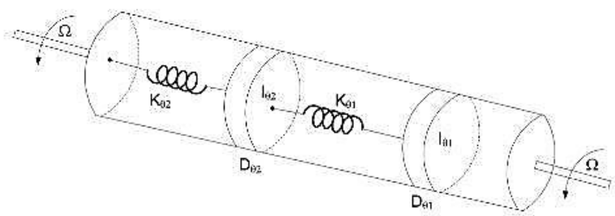
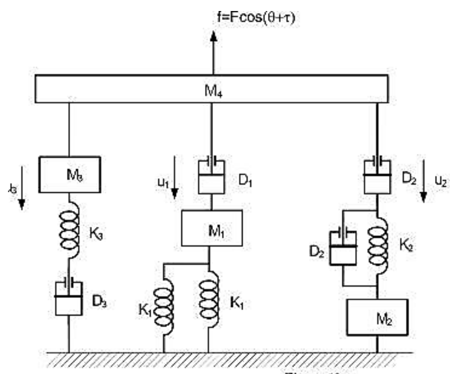


Figure 12.b



4.2- ELECTRICAL SYSTEMS.

4.2.1 Components of an electrical system: There are three basic elements in an electrical system,

i.e. (a) resistor (R), (b) inductor (L) and (c) capacitor (C). Electrical systems are of two types,

i.e. (i) voltage source electrical system

(ii) current source electrical system.

4.2.2. Voltage source electrical system:

* If i is the current through a resistor and v is the voltage drop in it, then

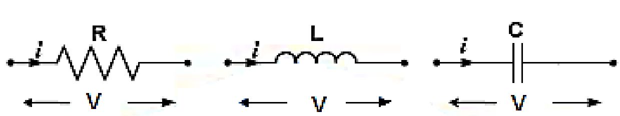
$$v = R\, i \ .$$

* If **i** is the current through an inductor and V is the voltage developed in it, then

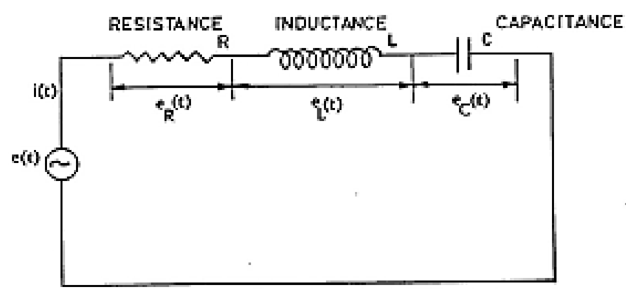
$$V = L\, di / dt \ .$$

* If i is the current through a capacitor and V is the voltage developed in it, then

$$V = 1/C \int i\, dt \ .$$



Example 5 Find system transfer function between function between the capacitance voltage to the source voltage in the following RLC circuit as shown in Fig. below .



Solution :-

- Voltage across the Resistance, $e_R(t) = iR$
- Voltage across the Inductance, $e_L(t) = L\, di/dt$
- Voltage across the capacitance, $e_c(t) = 1/C \int i dt$
- Total voltage, $e(t) = iR + L\, di/dt + 1/C \int i dt$

Laplace transform of the voltage source, $E(s) = I(s)\,[R + LS + 1/CS]$

Transfer function between output current and source voltage

$$\frac{I(s)}{E(s)} = \frac{1}{R + LS + \frac{1}{CS}}$$

Transfer function between capacitance voltage and source voltage

$$\frac{E_C(s)}{E(s)} = \frac{1}{CS(R + LS + \frac{1}{CS})} = \frac{1}{LCS^2 + CRS + 1} = \frac{\frac{1}{LC}}{S^2 + \frac{R}{L}S + \frac{1}{LC}} = \frac{\omega_n^2}{S^2 + 2\zeta\omega + \omega_n^2}$$

$$\omega_n^2 = \frac{1}{LC} \quad and \quad 2\zeta\omega_n = \frac{R}{L}$$

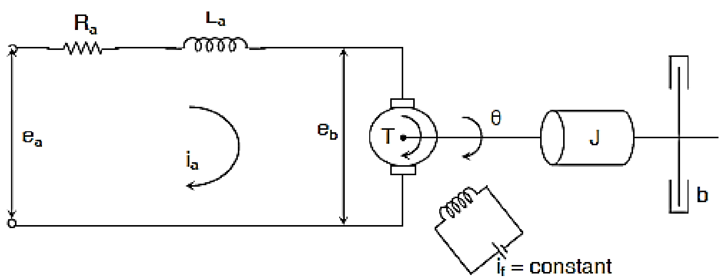
4.3- .Mathematical model of controlled DC Servomotors

The dc motors have separately excited fields. They are either armature-controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals.

4.3.1- Armature-Controlled dc motors.

The transfer function of an armature controlled d.c. motor relating angular shift in the shaft and the input armature voltage is derived here under. The circuit diagram of an armature controlled d.c. motor used for control systems is shown in Fig. below.

- The following components are used in armature-controlled dc motor



R_a = armature resistance (Ω)

L_a = armature inductance (H)

i_a = armature current (A)

i_f = field current (A)

e_a = applied armature voltage (V)

e_b = back emf (V).

θ = angular displacement of motor shaft (radian).

T = Torque by motor (N.m).

J = Equivalent moment of inertia of the motor and load referred to motor shaft (kg .m2)

b = Equivalent viscous friction coefficient (N.m/rad/sec).

- The torque is directly proportional to product of armature current .

$$T = K.\, i_a$$

- When armature is rotating, voltage proportional to the product of flux and angular velocity is induced in armature. But flux is constant.

$$e_b = K_b \,(d\theta / dt) \quad (e_b = \text{back emf and Kb : back emf constant}) \quad \text{--- 1}$$

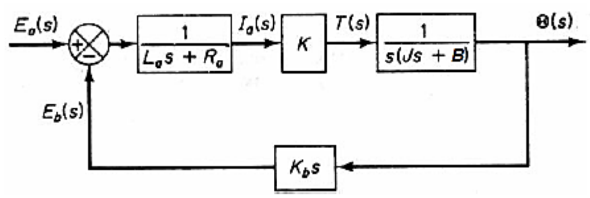
- The differential equation for armature circuit is,

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \quad \text{------------------------2}$$

- The torque equation can be given as.

$$T = J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = K \cdot i_a \qquad \text{------------------------ } 3$$

By taking the Laplace transform of above equations (1), (2), (3) then block diagram can be constructed as shown below.

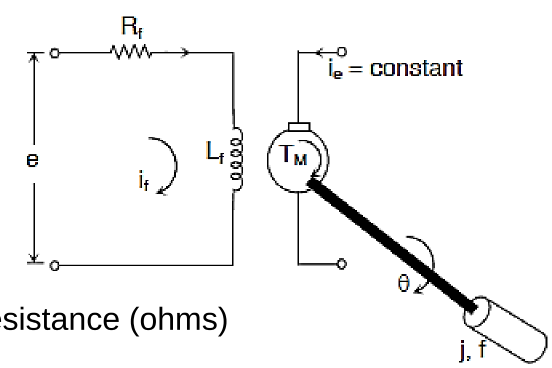


The transfer function is given as

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s\left[L_a J s^2 + (L_a B + R_a J)s + R_a B + K K_b\right]}$$

4.3.2- Field Controlled D.C. Motor

Field Controlled d.c. motor is shown in figure (a) & (b) below.



In this system,

R_f = Field winding resistance (ohms)

L_f = field winding inductance (henrys)

e = field control voltages (volts)

i_f = field current (amperes)

T_M = torque developed by motor (newton-m)

J = equivalent moment of inertia of motor and load referred to motor shaft (kg-m2)

B = equivalent viscous friction coefficient of motor and load referred to motor shaft .

θ = angular displacement of motor shaft (rad)

In the field controlled motor, the armature current is fed from a constant current source.

Therefore, $\mathbf{T_m = k_2 i_f}$ where k_2 is motor constant.

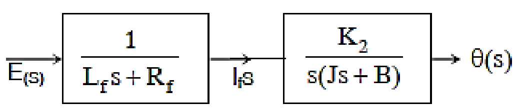
The equation for the field circuit is : $L_f \frac{di_f}{dt} + R_f i_f = e$ ------------------ 1

The torque equation is : $J \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T_M = k_r{'}\ i_f$ ----------------- 2

Taking the Laplace transform of equations (1) & (2). Assuming zero initial conditions, we get

$$(L_f s + R_f)\, I_f(s) = E(s)$$
$$(J s^2 + Bs)\, \theta(s) = T_M(s) = k_r{'}\, I f(s)$$

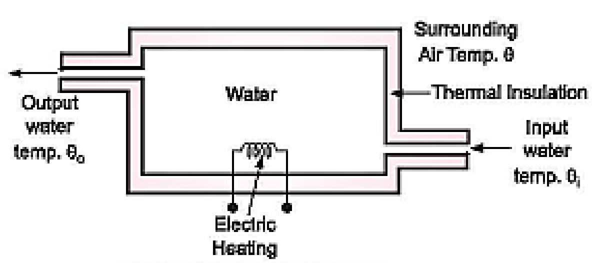
From the above equations, the transfer function of the motor is obtained as :



$$\frac{\theta(s)}{E(s)} = \frac{K_r{'}}{s(L_f s + R_f) \quad (Js + B)}$$

4.4- THERMAL SYSTEM

Heat transfer system. A thermal system used for heating flow of water is shown in Fig. below. An electric heating element is provided in the storage tank to heat the flow of water. The inlet water temperature is T_i and the outlet water temperature is T_o• The storage tank is insulated from surrounding atmosphere to reduce heat loss. The thermal resistance of the insulation is R (°C/joule/sec) .



Thermal water heating system.

If the thermal capacity is C (joulel°C) then the rate of heat flow for the water in the tank is given by $q_i = C\,\frac{dT}{dt}$

The rate of heat flow from the water to the surrounding atmosphere is

$$q_t = \frac{T_o - T}{R}$$

According to heat transfer principle following relation is obtained :

$$q = q_i + q_t = C\,\frac{dT}{dt} + \frac{T_o - T}{R} = C\,\frac{dT}{dt} + \frac{T_O}{R} - \frac{T}{R}$$

The variation of the water temperature T_0 is over and above the ambient temperature, i.e. T, therefore, the differential equation for the system is rewritten below neglecting the term T/R from the equation above ,we get

$$q = C\,\frac{dT}{dt} + \frac{T_O}{R}$$

In order to find the transfer function relating input heat energy q and the water temperature T_0 as output, take Laplace transform the both sides of the equation above . assuming initial condition as zero, therefore,

$$\frac{T_o(s)}{Q(s)} = \frac{R}{RCS + 1}$$

4.5- Liquid- Level Systems

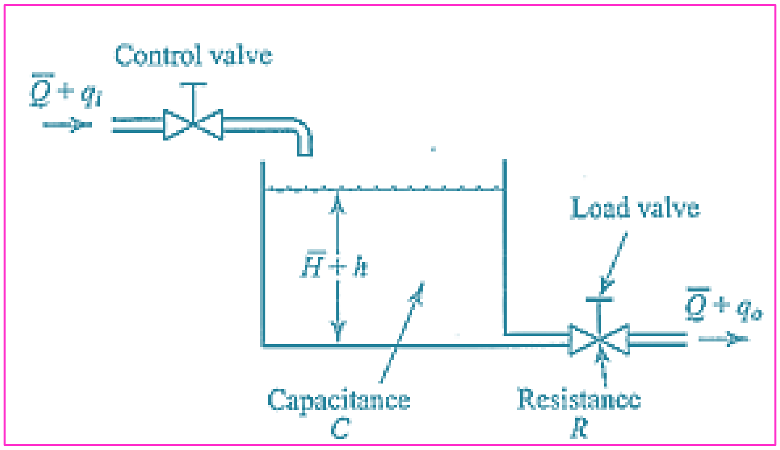
It is convenient to introduce the concept of **resistance and capacitance** to describe dynamic characteristics of liquid-level systems.

Consider the flow through a short pipe connecting two tanks. The resistance *R* for liquid flow in such a restriction is defined as the change in the level difference (the difference of the liquid levels of the two tanks) necessary to cause a unit change in flow rate;

R = Change in flow rate(m /s) /Change in level difference(m)

The relationship between the **flow rate and level difference** differs for the laminar flow and turbulent flow. The flow regimes are divided into laminar flow and turbulent flow, according to the magnitude of the Reynolds number.

Consider the liquid-level system shown below.



- For **laminar** flow, the resistance *R* is obtained as

$$R_1 = dQ / dH \quad = Q/H$$

The laminar-flow, resistance is constant and is analogous to the electrical resistance.

- If flow through the restriction is **turbulent**, the steady-state liquid flow rate is given by, $Q = K\sqrt{H}$

The value of the turbulent flow resistance R_t depends upon the flow rate and head.

- Linearization can be applied, provided that changes in the head and flow rate from their respective steady-state values are small.

1-In many cases, the value of the coefficient *K*, which depends upon the flow coefficient and the area of restriction, is not known. Then the resistance may be determined by plotting the head versus flow rate curve based on experimental data and measuring the slope of the curve at

2- The capacitance C of a tank 1 defined to be the change in quantity of **stored liquid** necessary to cause a unit change in the **potential (head).**

C = Change in head, m / Change in liquid stored, m

- The differential equation of the tank system shown can be obtained based on the assumption that the system is either linear or linearized.

- Consider a small disturbance from certain steady state operating point where Q and H are steady state flow rate and head, respectively.
- A small deviation in inflow q_i will cause a small deviation in out flow q_o and

head h. Since the inflow minus outflow during the small time interval dt is equal to the additional amount stored in the tank, then

$$C\, dh/dt = (q_i - q_o)$$

From the definition of resistance, the relationship between q_o and h is given by

$$q_o = h / R$$

The differential equation for this system for a constant value of R becomes

$$RC\,(dh/dt) + h = Rq_i$$

Note that RC is the time constant of the system. Taking Laplace transform of both sides of the differential equation we obtain the transfer function of the system

$$(RCS + 1)H(s) = R\, Q_i(s)$$

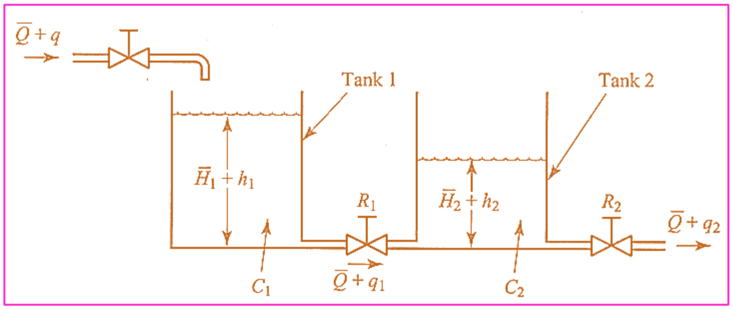
$$\frac{H(s)}{Qi(s)} = \frac{R}{(RCS+1)}$$

If q_o is taken as the output with qi as the input, then

$$\frac{Qo(s)}{Qi(s)} = \frac{1}{(RCS+1)}$$ where $H(s)=Q_o(s)*R$

4.5.1- Liquid-Level System with Interaction

If there is more than one tank, then the dynamics of the two (or more) tanks will interact. The transfer function is NOT the PRODUCT of the two first-order transfer functions. For the two tanks system shown, assume only small variations of the variables from the steady-state values. Using the symbols



Q_s = steady state flow rate, m_3/sec

q = incremental input flow rate from steady state value in the first tank, m_3/sec

q_1 = incremental output flow rate from steady state value in the first tank, m_3/sec

q_2 = incremental output flow rate from steady state value in the second tank, m_3/sec

H_{s1} = steady state head in the first tank, m

h_1 = incremental change of head in the first tank, m

H_{s2} = steady state head in the second tank, m

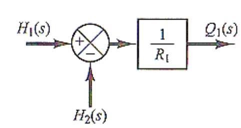
h_2 = incremental change of head in the second tank, m.

C_1, C_2 = Capacitance of the first and second tank, respectively, m^2

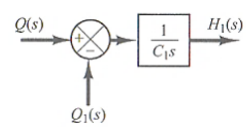
R_1, R_2 = resistance to output flow rate in the first and second tank, respectively, m/(m_3/sec)

For tank 1

$(h1-h2)*1/R_1=q_1$ by take Laplace transform $Q_1(s)=[H_1(s)-H2(s)]*1/R_1$

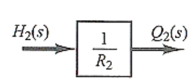


$C_1 dh_1/dt=q-q_1$ L.T. we get $H_1(s)=[Q(s)-Q_1(s)]*1/C_1s$

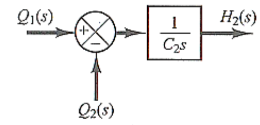


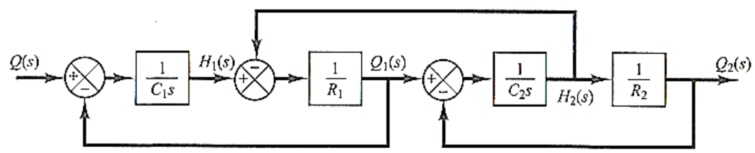
For tank II

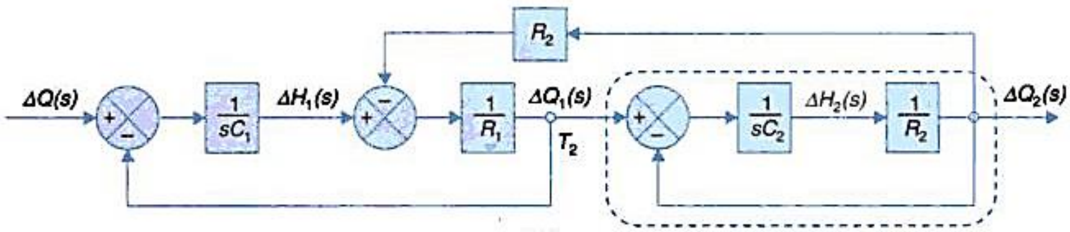
$h_2/R_2=q_2$ L.T. we get $Q_2(s)=(1/R_2)H_2(s)$



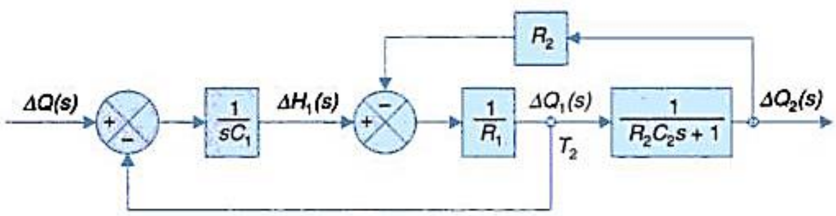
$C_2\,(dh_2/dt)=q_1-q_2$ L.T. we get $H_2(s)=[Q_1(s)-Q_2(s)](1/C_2S)$



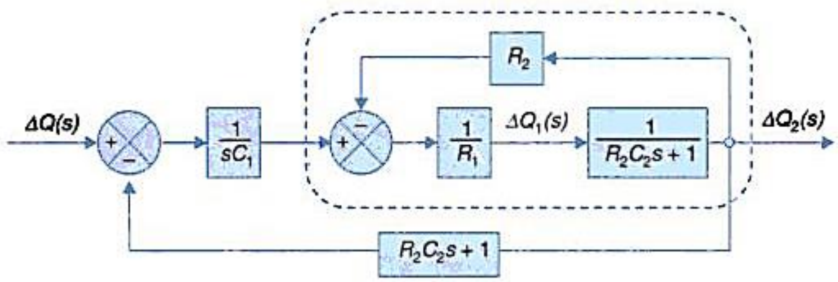




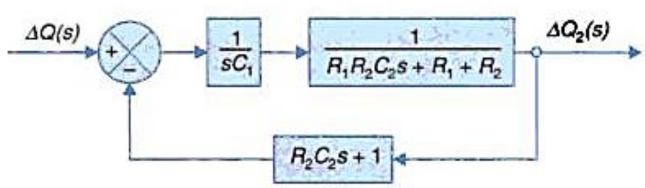
(h)

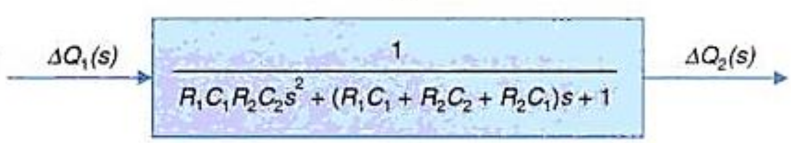


(i)



(j)



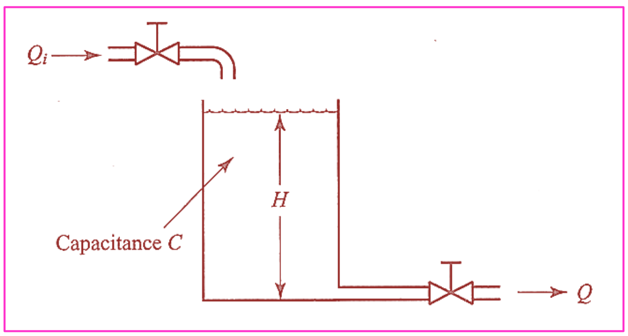


(l)

Example.5

In the liquid-level system shown in Figure, assume that the outlet flow rate Q (m^3/sec) through the out-flow valve is related to the head *H* (m) by $Q = K\sqrt{H} = 0.01\sqrt{H}$

Assume also that when the inflow rate Qi is 0.015 m^3/sec the head stays constant. For $t<0$ the system is at steady state ($Q_i=0.015\ m^3/sec$). At $t=0$ the inflow valve is closed and so there is no inflow for $t\geq 0$. Find the time necessary to empty the tank to half the original head. The capacitance *C* of the tank is 2 m^2.



Solution;-

When the head is stationary, the inflow rate equals the outflow rate. Thus head H_o at time $t=0$ can obtained as

$$Q=0.015= 0.01\sqrt{Ho} \Rightarrow Ho=2.25\ m$$

The equation for the system for $t>0$ is

$$-CdH=Qdt \qquad \Rightarrow \qquad dHdt=-Q/C= -0.01\sqrt{H}/2$$

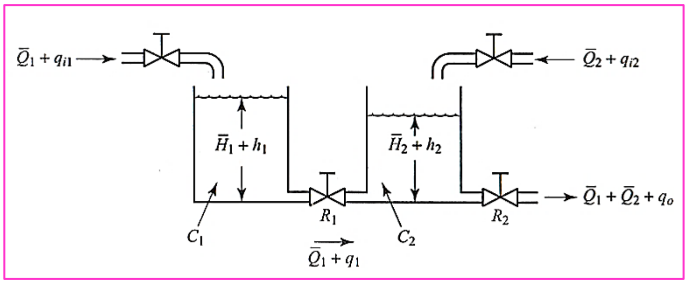
$$\text{Or} \quad dH/\sqrt{H}= -0.005\ dt$$

Then by integrating both sides of this equation $\int_{2.25}^{1.125} {dH}/{\sqrt{H}} = \int_0^{t1} -0.005dt$

$$2\sqrt{H}|_{2.25}^{1.125} =-0.005\ t_1 \Rightarrow t_1=175.7\ sec$$

Example.6

Consider the liquid-level system shown in Figure. In the system, $\bar{Q_1}$ and $\bar{Q_2}$ are steady-state inflow rates and $\bar{H_1}$ and $\bar{H_2}$ are steady-state heads. The quantities q_{i1}, q_{i2}, h_1, h_2 , and q_o are considered small. Obtain the dynamic equations of the system

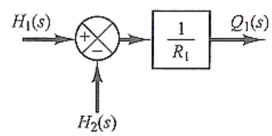


Solution ;-

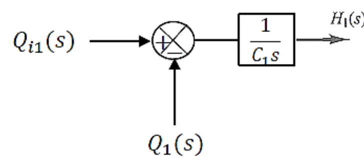
The equations for the system are

For tank I

$(h_1 - h_2)/R_1 = q_1 \quad \Rightarrow \quad Q_1(s) = [H_1(s) - H_2(s)] \,/R_1 \quad \Rightarrow$

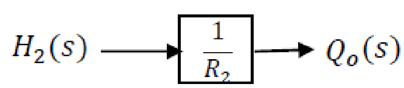


$C_1\,(dh/dt) = q_{i1} - q_1 \quad \Rightarrow H_1(s) = [Q_{i1}(s) - Q_1(s)]\ 1/C_1S$

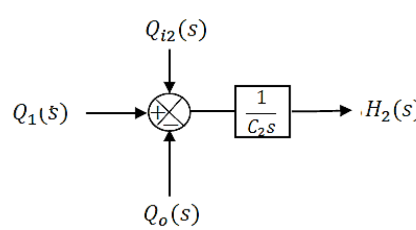


For tank2

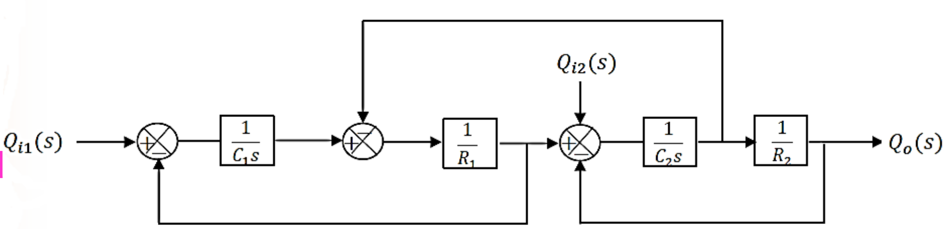
$h_2/R_2 = q_o \quad \Rightarrow \quad Qo(s) = (1/R_2\,)H_2(s) \quad \Rightarrow$



$C_2 dh_2/dt = q_1 + q_{i2} - q_o \quad \Rightarrow \quad H_2(s) = [Q_1(s) + Q_{i2}(s) - Q_o(s)]\ 1/C_2S$



The overall block diagram of the system is



4.6- Hyd

A schematic diagram of a hydraulic amplifier is shown in figure below . The position of the valve is designated by (x), and the position of large piston which moves the load is (y). When the valve is moved upward, the supply pressure admit oil to the upper side of the piston, and the fluid in the lower side of the piston is returned to the drain. For the reverse process, the valve is moved downward so that the supply pressure is connected to the bottom side of the big piston.

- When the mass of the load is negligible, the pressure drop across the valve remains constant. For this case, the rate of flow is proportional to the distance (x). Thus:

$$q(t) = C_1\, x(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

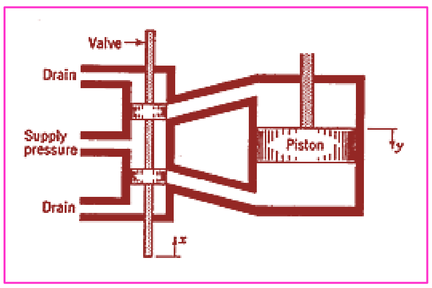
Where (q) is the rate of flow through the valve, also this rate of flow (q) is equal to the rate of change of volume of the chamber

$$q(t) = ASy(t) \ \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

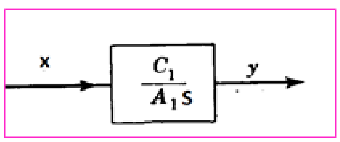
By equating (1) and (2), we get:

$$y(t) = (C_1/AS)x(t)$$

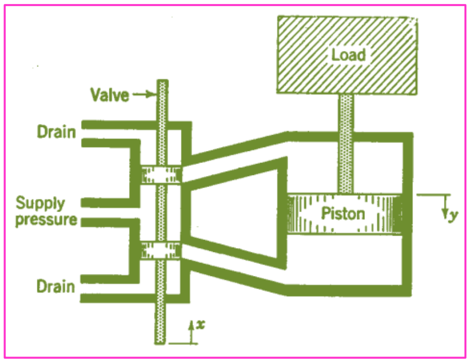
The (1/S) indicate that this hydraulic valve and piston combination in effect integrates hydraulically.



The block diagram representation of this hydraulic integrator is shown below .



- When the mass of the load effect as shown in figure below , The rate of flow (Q) to the cylinder is a function of the piston valve (X) and the pressure drop (P), that is:



$q= C_1x - C_2p$ ……………… (3)

The force transmitted to the load by the power piston is equal to the product of the pressure drop across the piston and the cross-sectional area (A) of the piston, thus:

$PA= Md^2y/dt^2 = MS^2Y(s)$ …………………………… (4)

Substitution of P from equation (3) into (4), gives:

$Q(s) = C_1X(s) - (C_2M/A)S^2 Y(s)$ But : $Q(s) = ASY(s)$

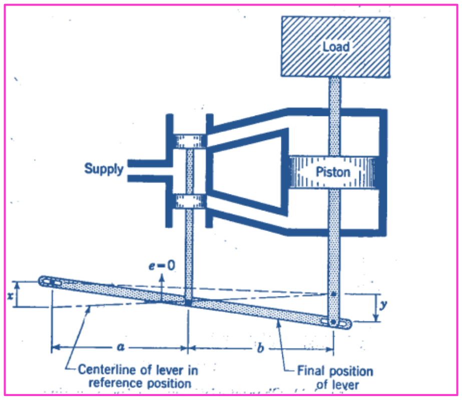
$\therefore y = (C_1A)\, X(s)/S(1+\tau S)$

$$\frac{Y(s)}{X(s)} = \frac{\frac{C_1}{A}}{s(1+\tau s)}$$

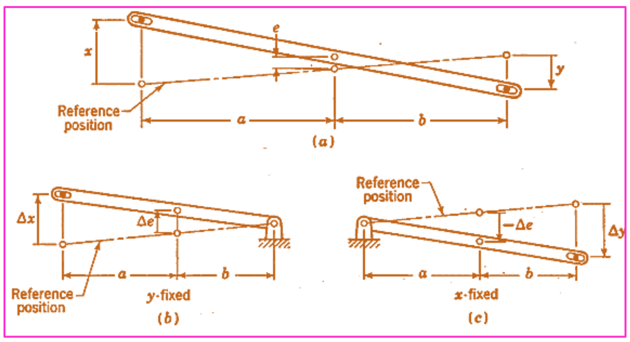
where: (τ) *is time constant is equal to* $=C_2/MA^2$.

4.6.1- Hydraulic Servomotor

A hydraulic servomotor is shown in figure below . A linkage called a ***walking beam*** connects the input position (x), the valve position (e) and the piston position (y). The operation of this servomotor may be visualized as follows: when the input (x) is changed from the reference position,



the walking beam first pivots about the connection at (y) because the large forces acting on the piston hold it in place temporarily. The relationship between the input (x) and the output (y) is shown in figure (2-a):



$$y/b = x/a \qquad \text{or} \qquad y = (b/a)x$$

Figure (2-b) illustrate the linkage with (y) fixed in the reference position. From similar triangles:

$$e_x = b/(a+b)\ x$$

Similarly, from figure (2-c), in which (X) is fixed in the reference position:

$$e_y = -\ a/(a+b)\ y$$

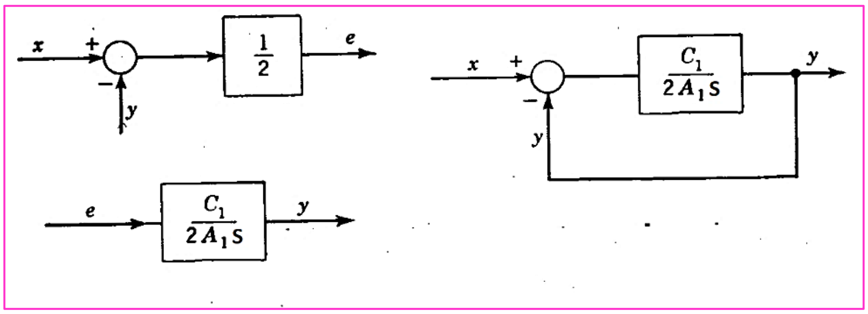
The minus sign indicates that (e) decreases as (y) increases. The overall walking beam linkage movement is:

$$e = (b/a+b)e_x - (a/a+b)e_y$$

For the case in which a = b , $e = (x-y)/2$

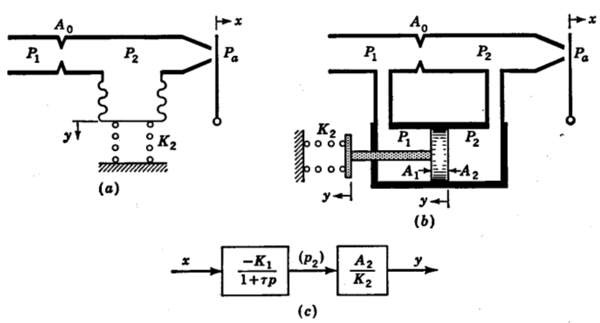
When mass of the load is negligible The equation for the valve and piston combination is given by: $y(t) = C_1 AS\, e(t)$

The block diagram representation for the preceding is shown in figure above



4.7 - Pneumatic Systems

A flapper valve, as shown in figure below (a), is one in which small changes in the position (X) of the flapper cause large variations in the controlled pressure (P_2) in the chamber. When the flapper is closed off, the pressure (P_2) in the chamber is equal to the supply pressure (P_1). If the flapper is opened wide, the chamber pressure approaches the ambient pressure (Pa).



For constant supply pressure (P_1) and fixed inlet orifice, the mass rate of flow in to the chamber (Min) is a function of the chamber pressure (P_2) only. Linearization gives:

$$min = -C_1 p_2$$

The minus sign indicates that as (P_2) increases, (m_{in}) decreases. Also, the mass rate of flow out from the chamber (M_o) is a function of (X) and (P_2). Thus:

$$mo = \partial M_o/\partial X|_i\ x + \partial M_o/\partial P_2|_i\ p_2 = C_2 x + C_3 p_2$$

The change in mass (w) of air in the chamber is the integral of ($m_{in} – m_o$):

$$w = (m_{in} - m_o)/S = [-C_1 p_2 - (C_2 x + C_3 p_2)]\ /S$$

Multiplying through by S shows that:

$$-C_1 p_2 + C_2 x - C_3 p_2 = S\ w$$

From the equation of state, the total mass W of air in the chamber is:

$$W = P_2 V_2 / R T_2$$

Where (V_2) is the volume of the chamber and (T_2) is the stagnation temperature of air in the chamber. Linearization yields for the change in mass (w) of air in the chamber:

$w = \partial W/\partial V_2|_i\ v_2 + \partial W/\partial P_2|_i\ p_2 = C_4 v_2 + C_5 p_2$

The change in volume of the chamber is equal to the area (A_2) times the change in position:

$$v_2 = A_2 y$$

The summation of forces acting on the bellows is:

$$P_2 A_2 = K_2 Y$$

The summation of forces acting on the piston is:

$$P_2 A_2 - P_1 A_1 = K_2$$

Because (P1A1) is constant, linearization of either of the preceding gives the same result. That is:

$$y = (A_2\ /K_2)\ P_2$$

Substitution (w) from equation (1) in to equation (2), then using the preceding expression to eliminate (v2) and (y):

$$p_2 = -\frac{K_1}{1 + \tau D} x$$

Where ($K_1 = \frac{C_2}{C_1 + C_3}$) and ($\tau = \frac{C_5 + A_2{}^2 C_4/K_2}{C_1 + C_3}$)

Because ($y = (A_2/K_2)\ p2$), it follows that:

$$y = \frac{A_2(-K_1)}{K_2(1 + \tau D)} x$$

4.8- Sensors and Transducers in Control Systems

- Transducers

It is an electronic device that converts physical quantities (energy, force, torque, light, motion,
position, microphones, pressure, air humidity, temperature, etc.) to an electrical signal.

Examples:

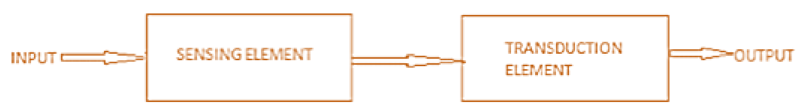
Thermocouple that changes temperature differences into a small voltage.
linear variable differential transformer (LVDT) used to measure displacement.
Loudspeakers, microphones, position, thermometers, antenna & pressure sensor, etc.

- Sensor:

Senses physical quantities and converts them into a readable form. For example, in a mercury thermometer, the mercury simply expands when the temperature rises to give a
reading for the user.
Both sensor and transducer are received and respond to a signal or stimulus from a physical
system. It produces a signal, which represents information about the system, which is used by
some type of telemetry, information, or control system.

- Is the transducer a sensor?

They are surrounded by or an object they are attached to, but, a sensor will give an output in
the same format and a transducer will convert the measurement into an electrical signal.

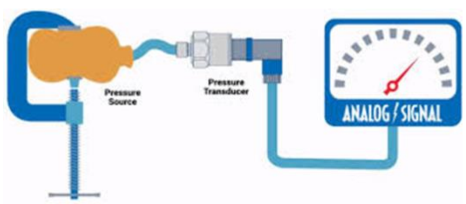


- Types of Transducers

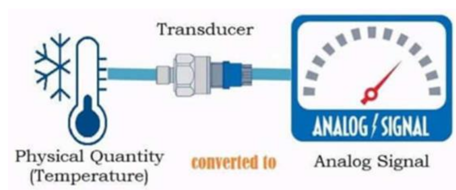
Transducers are classified into the following different types:

Transducers are named as level, pressure transducers, etc., just like sensors, on the basis of a physical parameters to be measured.

Transducers are also divided into current transducers or voltage transducers based on the output signal generated by transducers.



Pressure Transducer

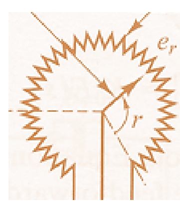


A Temperature Transducer

4.8.1 – Potentiometer

A potentiometer circuit can be used for comparing and measuring potential differences

The input to the device is in the form of mechanical displacement, either linear or rotation.



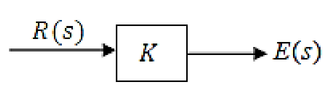
If ei is applied voltage $e_r = K_p\, r$

Where

e_r :- is output voltage

r :- shaft position

K_p :- potentiometer constant (volt/rad)



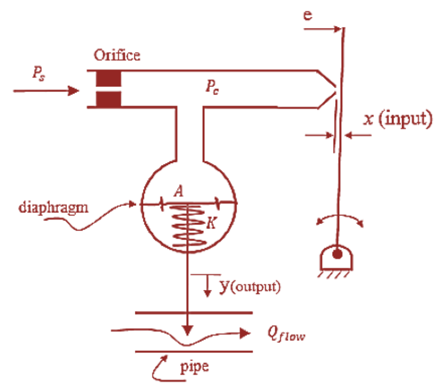
4.8.2- Tachometer

The tachometer is a sensor device used to measure the rotation speed of an object such as the engine shaft in a car. This device works essentially as a voltage generator, with the output voltage proportional to the magnitude of the angular velocity of the input shaft, or

$$e_t = K_T\, d\theta/\, dt \quad = K_T\, \omega$$

Where K_T:- is tachometer constant ($volt/(rad/sec)$)

4.8.3- Pneumatic valve:



The flow resistance through the orifice is:

q =ΔP R = (Ps−Pc)/ R ----- (1)

Pc = C1x ------(2)

The pressure inside the pneumatic system change w.r.t ti

C $\dot{P}_c$ = q =(Ps−Pc)/ R ---- (3)

R C $\dot{P}_c$ + Pc = Ps

Take Laplace transformation, we get:

$P_c / P_s = 1 / (RC\,s+1)$

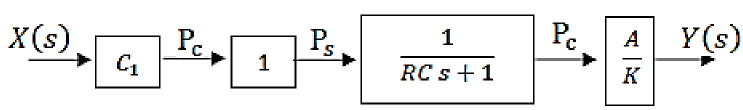
From diaphragm force balance

P_c A = K y(t) ------ (4)

Q flow = C_4 y(t) ------ (5)

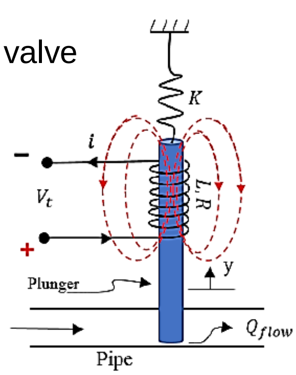
C_4 : valve constant ($m^2/sec.$)

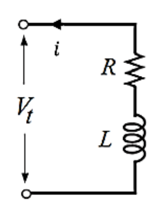
Assume x approach zero, This means that the valve is operating at its maximum capacity



4.8.4- Solenoid valve:

A solenoid valve is an electromechanically-operated valve





solenoid circuit

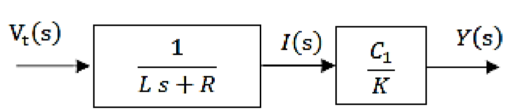
V_t (t) = R i(t) + L di/ dt -------------1

f(t) = C_1 i(t) -------------2

C_1: solenoid constant (N / amp.)

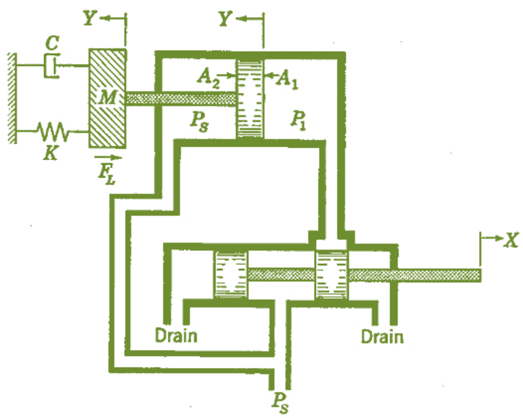
Force balance

f(t) = C1 i(t) = K y(t) ------------ 3

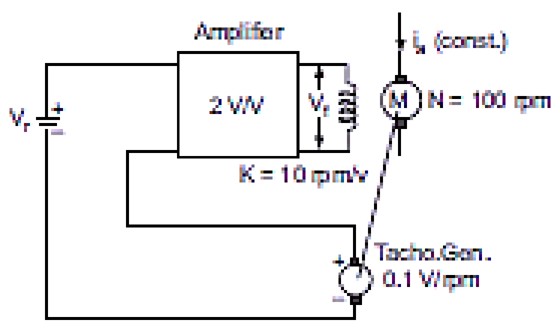


Home work .

Q.1- For the control system shown in the schematic diagram below, sketch a block diagram and simplify it to find out the overall transfer function.

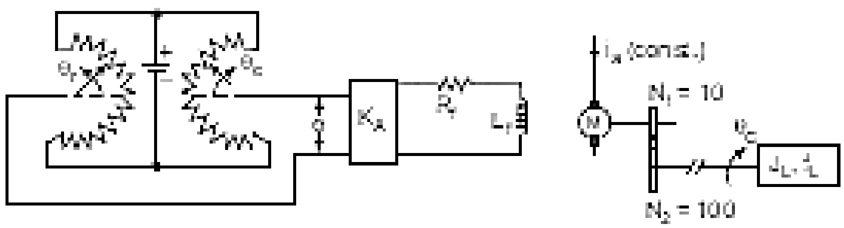


Q.2- The diagram given in Fig. below represents a closed-loop control system for regulating the speed of a field controlled d.c. motor. Determine the value of the reference voltage if the speed is to be maintained at 100 r.p.m.



Q.3- A torque controller uses a split field. d motor for position control. The error detector gain is 10 V per radian error and the amplifier trans conductance is 100 mA/V. The motor torque constant is 5 x 10^{-4} Nm/mA. The moment of inertia and the coefficient of viscous friction at the motor shaft are respectively 1.25 x 10^{-5} kg-m^2 and 5 x 10-4 Nm/(rad/ sec). The motor is coupled to the load through a gear having a ratio 20: 1.Draw the block diagram of the system showing the transfer function of each block and determine the overall transfer function relating the output and input.

Q.4- For the closed-loop control system shown in Fig. below. Draw the block diagram and determine the overall transfer function relating θ_c and θ_r.



The system has the following details:

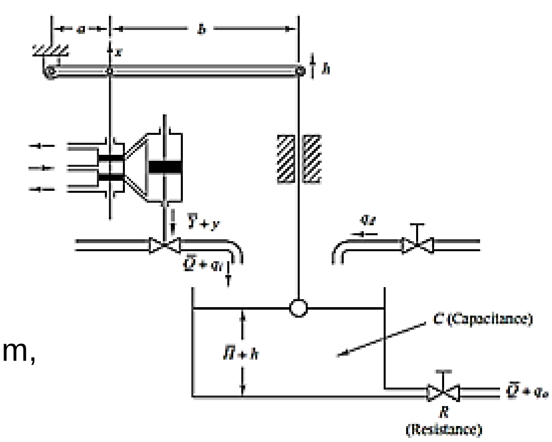
- Error detector gain K = 8 V/rad, amplifier gain K_A= 10 V/V, R_f= 5 ohm, L_f= 0.25 H,
- Motor torque constant K_m= 0.05 Nm/A. Moment of inertia for motors = 0.02 kg-m^2 .
- oefficient of viscous friction for motor f m = 0. 03 Nm/ (rad I sec).
- Moment of inertia for load J_L = 3.0 kg-m^2.
- Coefficient of viscous friction for load B_l = 4.5 Nm/(rad/ sec).

Q.5- Consider the liquid-level control system shown in Figure below. assume that the velocity of the power piston (valve) is proportional to pilot-valve displacement x,

or $dy/dt = K_1 x$

where K_1 is a positive constant. We also assume that the change in the inflow rate q_i is negatively proportional to the change in the valve opening y,

or $q_i = -K_v y$



Assuming the following numerical values for the system,

C=2 m^2, R= 0.5 sec/m^2, K_v=1 m^2/ sec

a=0.25 m, b= 0.75 m, K_1=4 sec^{-1}

Obtain the transfer function $H(s)/Q_d(s)$.

Q.6-Find the transfer function for the electro mechanic system shown in figure below .